DYNAMIC OPTIMISATION PROBLEMS:

DIFFERENT RESOLUTION METHODS REGARDING AGRICULTURE AND NATURAL RESOURCE ECONOMICS

Working Paper

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March 2002

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Abstract:

There are two well-known methods mathematically equivalent to solve stochastic dynamic optimisation problems concerning natural and agricultural resources: Stochastic Dynamic Programming (SDP) and Discrete Stochastic Programming (DSP). Both are subject to major limitations in practice.

Although the dynamic programming method permits considering a large number of decision stages, since the multi-stage problem in question is broken down into several one-stage problems, the number of state and decision variables must remain limited. In practice, this technique makes it necessary to limit the possible values of the model's state and decision variables to a finite discrete set. The solutions obtained are therefore approximate and, in the case of non-linear functions, the errors can be non-negligible. Dynamic programming models, which are much used for fisheries and forestry management, are difficult to adapt to agricultural resource problems.

As for discrete stochastic programming, it permits taking into account the diversity of activities and constraints specific to agricultural decision problems. DSP permits working with continuous variables and do not require the utility function to be separable. Nonetheless, application of this technique remains limited to problems with a low number of stages: since optimisation is inter-temporal, the model's size increases exponentially with the number of decision stages.

By making the hypothesis that the decision-maker is more «myopic» than the dynamic programming assumes, we shall propose another method of solving stochastic dynamic problems: recursive stochastic programming (RSP). The main difference of this method in comparison with the previous ones is the way information enters the model. In this case, information arrives step by step and then not all the information is available at the time of making decisions. The key idea is therefore that the decision-maker cannot fully anticipate the responses of nature and must opt for a sub-optimal decision. Since the stochastic dynamic problem is solved by a sequence of inter-temporal optimisations, this approach allows simultaneously considering a high number of variables and stages and permits to define separately short and long term decisions. It has major advantages when the system must be represented by a considerable number of state variables or in the case of a large number of possible activities (reservoir management, irrigation management, soil erosion, etc.).

In this paper, these different methods are compared and a practical application concerning water allocation in agriculture is presented, where levels of water applied to different crops are decided in the short term and investment in the long term.

Keywords: Stochastic programming, Recursive stochastic programming, Agricultural resources.

The authors thank Mabel Tidball for her insightful comments and suggestions. The authors wish also thank all the assistants to the seminar LAMETA (Montpellier, October 4, 2000) for their helpful comments on an earlier version of this paper.

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1. GENERAL INTRODUCTION

The need to take into account sustainability in agricultural resource management is now universally admitted. While the term «sustainability» can mean different things to different people, it always involves a consideration of the future. From an economic point of view, sustainability can be defined as an improvement of the performance of a system so as not to exhaust the basic natural resources on which its future performance depends (Pierce and al., 1990). This definition emphasizes the importance of preserving the natural resource base.

Thus, sustainability is a dynamic concept with underlying inter-temporal trade-offs. Implementing the notion of sustainability requires not only knowing to what extent short term profit is preferable to future profit, but also the effect of current production decisions on the future performance of the system.

From this standpoint, natural resources can be understood as stocks of natural capital. Regarding renewable resources, the availability of a resource will decrease if its extraction rate exceeds its rate of natural regeneration. For instance, if the water extraction rate from an aquifer exceeds the rate of replenishment, the availability of this resource will decrease over time.

Furthermore, most natural resource problems involve sequential, risky and irreversible decisions. Thus the problem of natural resource management is one of inter-temporal allocation in a context of uncertainty and irreversibility. The mathematical basis for solving these dynamic problems is provided by the optimal control theory. The analytical solution of optimal control models, however, is only possible in the case of very simple problems.

Thus, applied research calls for operational research techniques to treat increasingly complex resource management problems. Tools such as simulation, mathematical programming and dynamic programming can be used, depending on the problem's complexity.

Simulation models are pertinent for extremely non-linear systems containing stochastic elements. These models do not use optimisation algorithms and allows us to analyse the evolution of the system over time in different scenarios.

On the other hand, mathematical and dynamic programming models are inter-temporal optimisation models capable of obtaining an optimal solution, given the system's objective function and constraints. A reason often cited for the low adoption of these stochastic optimisation techniques is what Richard Bellman, the father of dynamic programming, termed the *curse of dimensionality*, which refers to the explosive growth of the model as the number of variables increases.

As we will show further on, dynamic programming models - well suited for fisheries and forestry management - are more difficult to apply to agricultural resource problems.

In this paper, these different techniques for solving dynamic optimisation problems are compared, particularly mathematical and dynamic programming. We emphasize the advantages and disadvantages of these methods and their respective fields of application. Furthermore, we propose an alternative technique for solving stochastic dynamic problems.

2. CLASSIFICATION OF DYNAMIC PROBLEMS

2.1. Time and models

Agricultural and resource economics models are often constrained optimisation problems. The general form of such a problem is:

$$\text{Optimise} \quad f\left(x_1, x_2, \dots, x_n\right)$$
$$\text{subject to} \quad g_i\left(x_1, x_2, \dots, x_n\right) = 0$$

where $f(\cdot)$ is a given *objective function* of *n decision variables* $x_1, \dots, x_n$ which must obey the constraint set given by equations g_i.

Regarding time representation, we differentiate two types of models:

- **Static models** do not take explicit account of time. Decision variables does not depend on time. Calculations are carried out to obtain the optimal value of the objective function at a given moment. Time is not explicitly included in the model's structure.
- **Dynamic models** take time into account explicitly. Some of the decision variables are functions of time (usually separated into state variables and control variables). Model solution gives optimal decisions over time. The temporal element is taken into account in different ways described further on. Here, however, we provide a temporary, "fuzzy" definition, in order to introduce additional information later.

The full advantages of dynamic analysis are apparent when analysing problems in which certain decisions have consequences on several future periods or when the problem consists in analysing the transition of a system from one state to another one over time.

Below we present a classification of dynamic models. Provided that it does not exist an universally accepted terminology, we have attempted to use the terms most often employed in the literature.

Dynamic models

- Inter-temporal optimisation models: the model takes into account all the periods included in the *planning horizon.*
  - Deterministic models
  - Stochastic models
    - Single decision models
    - Sequential decision models - representation of sequential decision-making with the gradual incorporation of information.
- Recursive models – sequential optimisation, each optimisation depends on the results of the previous iteration.

2.2. Inter-temporal dynamic optimisation models

2.2.1. Introduction

A typical case in which dynamic analysis is very pertinent is that of investments. A capital good with a lifetime of several years can be taken into account in an annual static model by considering its annual rental value as a cost linked to its use. This may suffice if the problem to be analysed does not require thorough study of the investment process. However, building a dynamic inter-temporal optimisation model is pertinent if the investment through time must be analysed in detail in order to study, for example, the impact of different credit policies.

In this type of model, optimisation is performed over a discounted flow of returns (net revenue, consumption, …). The choice depends on the type of problem analysed. Optimisation is therefore inter-temporal and the period of time considered in the analysis is termed *planning horizon*. Formulating and using these models lead to different problems. As temporal preference must be taken into account for optimisation over several stages (generally yearly periods), the choice of the discount rate is a difficult and often controversial issue.

2.2.2. Deterministic dynamic models

Deterministic dynamic models contain complete and perfect information on the future. All the model parameters, such as future prices, climate, yields, etc., are supposed «known» by the decision-maker. This type of model optimises decision-making that takes into account future costs and profits, by inter-temporal arbitration.

2.2.3. Stochastic dynamic models

2.2.3.1. Single decision stochastic models

These models contain knowledge of the future in terms of probabilities of states of nature and optimise a utility function. Uncertainty can be about some resource availability constraints or/and the objective function coefficients. Here, the usual techniques to model risk in dynamic models (mean-variance, chance-constrained programming, etc.) can be used.

2.2.3.2. Sequential decision stochastic models

These models represent a sequential decision-making process. The information available to the model is introduced in steps, permitting progressive decision-making adjustment in the framework of a decision tree with branches at each decision step.

2.3. Recursive models

Recursive models (Day, 1961) are also dynamic models[1], since different decision stages are represented explicitly. The essential difference with inter-temporal optimisation models resides in their optimisation method. Rather than performing optimisation over the entire planning horizon, *it is performed for each stage individually, though the results of stage t will influence the initial data in stage t + 1.* Day originally developed this method to describe the process of adjustment between a real situation and an optimal one obtained after optimisation, with the aim of describing gradual adaptation to changes in exogenous parameters, by using what he called "flexibility constraints". The procedure was simple: given a certain level of rotation, for example, constraints are introduced that permit a difference of more or less 5% from this initial rotation. Repeating these constraints over several iterations leads the model to a situation in which they are no longer active, insofar as the initial rotation is reinitialised in each

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[1] Some authors consider only inter-temporal optimisation models as dynamic, though since an increasing number of recursive models are termed dynamic, we have chosen to consider them as belonging to the family of dynamic models.

iteration. Thus this is a way of introducing gradual and flexible adaptation towards a new equilibrium.

Searching for market equilibrium with time lags (cobweb model) is a typical example of the recursive procedure.

Many models called "dynamic" are actually recursive models, as defined here. This is so with dynamic general equilibrium models whose solution for one year is used to reinitialise the parameters used by the model the following year. In some agricultural sector models, the recursive procedure is used to define the equilibrium price and treasury flows between periods[2].

It is also possible to build a recursive model which, for each iteration, is composed of inter-temporal optimisation models, i.e. *it is a sequence of dynamic models reinitialised after each optimisation by incorporating additional information*. This information can be partially composed of the results of the first optimisation and be partially exogenous. Once the model has been reinitialised, it runs with a new planning horizon shifted by one stage.

3. DYNAMIC OPTIMISATION PROBLEMS: RESOLUTION METHODS

Here, we deal with the different methods of solving inter-temporal optimisation problems, giving emphasis to dynamic programming and mathematical programming techniques. Deterministic dynamic models and dynamic single decision stochastic models, as defined in the previous section, can be solved using similar methods. Consequently, we have decided to present them together as non-sequential dynamic optimisation problems (section 3.1). In these types of model, the sequence of optimal decisions is determined at the beginning of the decision process and no modification is made afterwards.

In the dynamic sequential decision stochastic models presented in section 3.2, decisions are taken sequentially and the decision-maker can adjust them as and when he has additional information to enter.

2 This is the case of the CAPRI model (http://www.agp.uni-bonn.de/agpo/rsrch/capri/eaaecapri.pdf) and the MATA family of models (http://www.adelaide.edu.au/CIES/iwp9808.pdf).

3.1. The problem of non-sequential dynamic optimisation

As said previously, from a management standpoint, natural resources are better viewed of as stocks of natural capital that provide a flow of services (Wilen, 1985). Thus the resource allocation problem consists in maximising the benefit obtained from using flows of resources through time, taking into account that current use can influence future availability. *Therefore the problem of allocating natural resources is a dynamic problem*. Consequently, optimal control theory provides the correct approach to natural resource management.

In this paper analysis is limited to discrete time, finite horizon problems. These problems include a decision sequence through time and can be represented by a decision tree. Decisions taken for each stage influence the possible results for the following stages. This kind of optimisation is inter-temporal.

Consider a simple problem of optimal allocation of a natural resource in a dynamic framework. For each period of time t, the system is described by a *state variable* (x_t) and a *control variable* (u_t); the former represents the stock of the resource while the latter represents the extraction decision. Let us suppose that we have an initial quantity of resource (x_1), that the decisions on the use of the resource (u_t) are taken at the beginning of each decision stage t and that the profit obtained in each stage is given by $r_t(x_t,u_t)$.

The *objective function* to be maximised (profit or inter-temporal utility[3]) is generally expressed as a function of the first stage:

$$v_1(x_1,u_1,\ldots,u_T) = f\,\left[r_1(x_1,u_1), r_2(x_2,u_2),\ldots, r_T(x_T,u_T)\right] \tag{1}$$

Equation (1) expresses that the current value of the resource (v_1) is a function of the returns obtained throughout the planning horizon ($t = 1,\ldots,T$).

In a dynamic framework, the stock of the resource in year $t+1$ is a function of both the decisions taken in year t and the *autonomous* progression of the resource from t to $t+1$. This relation of dependence is expressed by the *equation of motion* or transition equation:

$$x_{t+1} - x_t = g_t(x_t,u_t)$$

To simplify the problem, we make a certain number of hypotheses:

[3] The problem of defining individual or social utility functions is extremely difficult and is not dealt with in this paper.

- the objective function is an additively separable function defined by the discounted sum of the returns obtained throughout the planning horizon, given that r is the discount factor;
- functions r_t and g_t are assumed to be continually differentiable to the order two; and
- the stock of the resource at the end of the planning horizon has a final value given by $F(x_{T+1})$.

The standard problem consists in determining the sequence of decisions (u_t) that maximise the objective function by respecting the constraints:

$$\text{Maximize} \quad \sum_{t=1}^{T} r^{\,t-1}\, r_t(x_t,u_t) \quad + \quad r^T\, F(x_{T+1}) \tag{2}$$

$$\text{subject to} \quad x_{t+1} - x_t = g_t(x_t,u_t) \qquad t = 1,2,\ldots,T \tag{3}$$

$$x(1) = x_1 \tag{4}$$

Therefore the problem consists in maximising the current value of the profits obtained throughout the planning horizon increased by the final value of the resource.

Equation (3) is the equation of motion or transition equation that reflects that the stock of the resource through time is both a function of resource extraction and resource renewal. Bio-economic models are spoken of in the case where biophysical models are used to obtain functions $g_t(x_t,u_t)$ or $r_t(x_t,u_t)$.

3.1.1. Analytical solution

The analytical solution of the natural resource management problem given by equations (2) to (4) calls on optimal control theory. The principle of maximum defines the inter-temporal optimality conditions of the optimal control problems.

Functions r_t and g_t are assumed to be continuous and differentiable to the order two. The search for the optimum is done by introducing the Hamiltonian:

$$H_t(x_t,u_t,l_{t+1}) = r_t(x_t,u_t) + r\; l_{t+1}\; g_t(x_t,u_t) \tag{5}$$

In the framework of natural resource economics, the Hamiltonian can be interpreted as the total profit resulting from the use of the resource: the first part represents the profit in the current stage (t), whereas the second part represents the change in the value of the stock. The multipliers l_{t+1} represent the values (measured in $t=1$) given to an additional unit of stock x_{t+1} in stage $t+1$.

Maximising the Hamiltonian for each stage t therefore amount to maximising the objective function. Introducing the Hamiltonian permits transforming the problem of constrained optimisation into one free of constraints.

The first-order conditions for profit maximisation are:

$$\frac{\partial H_t}{\partial u_t} = \frac{\partial r_t(\cdot)}{\partial u_t} + r\ l_{t+1}\frac{\partial g_t(\cdot)}{\partial u_t} = 0 \qquad t = 1,\ldots,T \tag{6}$$

$$r\ l_{t+1} - l_t = -\frac{\partial H(\cdot)}{\partial x_t} \qquad t = 2,\ldots,T \tag{7}$$

$$x_{t+1} - x_t = \frac{\partial H(\cdot)}{\partial (r\ l_{t+1})} = g_t(\cdot) \qquad t = 1,\ldots T \tag{8}$$

$$l_{T+1} = F'(\cdot) \tag{9}$$

$$x(1) = x_1 \tag{10}$$

In equation (6), the right hand term is divided into two parts: the former is the marginal profit from using the resource in the current stage, whereas the latter part reflects the influence of the decision taken u_t on the value of the resource over the remaining stages, i.e. the inter-temporal cost of resource extraction, or *user cost*. The shadow price l_{t+1} reflects the increase of profit throughout the remaining stages if the stock of the resource increases by one unit (or the loss of profit throughout the remaining planning horizon due to the consumption of an additional unit in the current stage).

Equation (7) indicates the change of Lagrange multipliers through time. Equation (8) is the transition equation while equations (9) and (10) are the boundary conditions (final and initial conditions).

Equations (6) to (10) form a system of (3T+1) equations with (3T+1) unknowns: u_t for $t = 1,\ldots T$; x_t for $t = 2,\ldots T+1$ and l_t for $t = 1,\ldots, T+1$. It is not always possible to solve this system of equations simultaneously. Although the theory of optimal control is well-suited for dealing with natural resource problems, if the extremes are not interior points, or the functions are not continuous and differentiable, no analytical solution is possible. In practice, it is usual to use resolution algorithms such as dynamic programming and mathematical programming.

These methods of solving dynamic optimisation problems will be presented in what follows, with emphasis being given to possibilities of applying each method in the field of natural resource economics rather than to the resolution procedure.

3.1.2. Method of dynamic programming

The dynamic programming method was developed by Richard Bellman during the 1950's. It permits solving this type of problem, provided that the objective function is separable.

By designating the optimal value of the resource stock in stage t by $V_t(x_t)$ – i.e. the value of the resource when optimal decisions u^*_t, $u^*_{t+1},\ldots,u^*_T$ have been taken – the problem consists in finding $V_1(x_1)$:

$$V_1(x_1) = \max_{u_1,u_2,\ldots,u_T} f\left[r_1(x_1,u_1), r_2(x_2,u_2),\ldots,r_T(x_T,u_T)\right] \tag{11}$$

If the objective function respects the conditions of separability, the multi-stage problem can be broken down into T one-stage problems by using the *recursive relation*:

$$V_t(x_t) = \max_{u_t} f\left[r_t(x_t,u_t),\ V_{t+1}(x_{t+1})\right] \qquad t = T, T-1,\ldots 1 \tag{12}$$

In this equation $V_t(x_t)$ represents the optimal value of the objective function throughout the remaining planning horizon under optimal decisions. If the objective function is the sum of the discounted profits of each stage, the problem consists in determining the optimal sequence of decisions $u^*_1,\ldots,u^*_T$ which obeys:

$$V_t(x_t) = max_{u_t}\left[r_t(x_t,u_t) + r\ \ V_{t+1}\left(x_t + g_t(x_t,u_t)\right)\right] \qquad t = T, T-1,\ldots,1 \tag{13}$$

$$V_{T+1}(x_{T+1}) = F(x_{T+1}) \tag{14}$$

$$x(1) = x_1 \tag{15}$$

Functional equation (13) permits determining $V(x_t)$ once $V(x_{t+1})$ is known. Since the final value is assumed as known, it is possible to determine the optimal value for stage T:

$$V_T(x_T) = max_{u_T}\left[r_T(x_T,u_T) + r\ \ F(x_{T+1})\right] \tag{16}$$

Solving this equation for each possible value of the state variable (x_T) allows us to obtain u^*_t and $V_T(x_T)$ and repeating this procedure for stages T-1, T-2,…, 1 permits solving the problem.

In practice, the analytical resolution of the recursive relation (13) demands that functions $V_t(x_t)$ and $r_t(x_t,u_t)$ be differentiable and that an interior solution exists. If these conditions are not respected, the problem can still be solved by using numerical methods. However, numerical resolution limits the possible values of the state and control variables to a discrete set for each stage t. Furthermore, any decision taken in stage t must lead to one of the possible values of x_{t+1}.

The numerical formulation of the problem can be interpreted as the search for an optimal path through a nodal network, since the characteristics of the optimal path are given by *Bellman's principle of optimality* (1957): «an optimal policy is one in which, whatever the initial state and initial decision, the following decisions must constitute an optimal policy in relation to the state resulting from the initial decision». The dynamic programming method permits obtaining a decision rule so that it is easy to determine the optimal trajectory for different initial conditions.

Numeric dynamic programming is a very flexible method that permits the resolution of inter-temporal optimisation problems even when functions r_t and g_t are not continuous and differentiable. It permits obtaining a full decision rule, whereas other techniques, such as the multi-stage programming methods seen further on, only give solutions for specific initial conditions.

Since the recursive relation must be solved for all the values related to the state variables, the main disadvantage of this model is that its size explodes when the number of state variables increases (the «curse of dimensionality»).

Many applications of this method exist in the field of agricultural and natural resource economics. Kennedy (1986) wrote a detailed review on this subject.

3.1.3. Mathematical programming method

The problem of dynamic optimisation can also be solved as a constrained optimisation problem, by using mathematical programming techniques.

With this method, state and control variables are defined as activities while the transition equations are defined as multi-period constraints that link the stages together. Mathematical programming permits obtaining an optimal solution, given the constraints and the objective function. Non-linear programming algorithms and techniques now exist that allow incorporating uncertainty not only in the objective function, but also in the constraints.

Whereas the dynamic programming method solves problems recursively, by backward induction, the mathematical programming method consists in solving all the following equations simultaneously, by using one of the existing algorithms:

$$\text{Maximize} \quad \sum_{t=1}^{T} r^{\,t-1}\, r_t(x_t, u_t) \quad + \quad r^{T}\, F(x_{T+1}) \tag{17}$$

subject to $x_{t+1} - x_t = g_t(x_t, u_t) \qquad t = 1,2,\ldots,T$ (18)

$x(1) = x_1$ (19)

The mathematical programming method permits incorporating in the model the diversity of activities and constraints specific to decision-making in agriculture. The advantages of this method are considerable in comparison to dynamic programming when the problem is deterministic or when stochastic components can be approached with the non-sequential techniques used in risk programming.

Although dynamic programming remains the technique used most often for solving dynamic optimisation problems, several authors emphasise the advantages of mathematical programming when incorporating the interdependencies between the different resource allocation decisions in the model (Standiford and Howitt, 1992; Yates and Rehman, 1998).

The mathematical programming method permits working with continuous variables and incorporating all the activities and constraints considered necessary. Nevertheless, it is not always possible to obtain a global maximum for very complex non-linear models. This difficulty could be overcome by using genetic algorithms (Cacho, 2000).

3.2. Problem of sequential dynamic optimisation

The problem becomes stochastic if the state variables and/or the results of each stage depend not only on the state of the system and the decisions taken, but also on random variables that the decision-maker cannot control.

Generally, a sequential stochastic decision problem can be represented by a decision tree. For example, Figure 1 represents a problem with three decision stages and two states of nature.

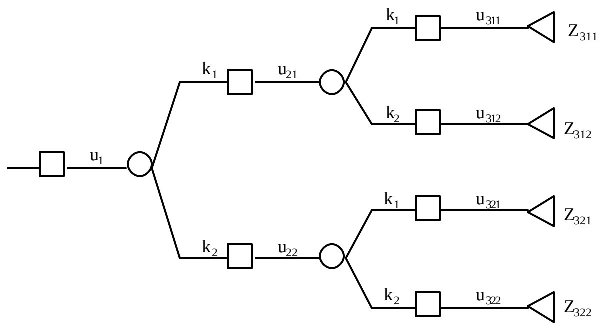

Figure 1. Decision tree (three decision stages and two states of nature).

The diagram shows that by starting from an initial state of the system (represented by a small square), the farmer takes decisions in stage 1 (u_1). Later, according to the state of nature occurring (k_1 or k_2), the farmer can take other decisions (u_{21} is, for example, the decisions taken in stage 2, taking into account the state of nature k_1).

In sequential stochastic problems, one of the objective functions used most frequently is the mathematical expectation of total discounted profit[4]:

$$v_1(x_1) = E\left[f \left\{ r_1(x_1,u_1,k_1), r_2(x_2,u_2,k_2), \ldots, r_T(x_T,u_T,k_T) \right\} \right] \qquad (20)$$

Let us suppose that random variables (k_t) take different discrete values in each stage t with associated probabilities $p_t(k_t)$, and that the objective function is the expected present value. By hypothesising that the problem can be written as a Markovian decision process, i.e. that the state of the system in stage $t+1$ only depends on x_t, u_t and k_t, the problem is written as:

$$\text{Maximize} \quad \sum_{t=1}^{T} r^{\,t-1} E\left[r_t(x_t,u_t,k_t) \right] + r^T F(x_{T+1}) \qquad (21)$$

$$\text{subject to} \quad x_{t+1} - x_t = g_t(x_t,u_t,k_t) \qquad t = 1,2,\ldots,T-1 \qquad (22)$$

$$x(1) = x_1 \qquad (23)$$

given that:

[4] In the case where the decision-maker is not considered risk-neutral, other objective functions can be proposed.

$$E\left[r_t(x_t, u_t, k_t)\right] = \sum_k p_t(k_t) \;\; r_t(x_t, u_t, k_t) \qquad (24)$$

As seen further on, this problem can be solved by using dynamic programming or discrete stochastic programming.

3.2.1. The stochastic dynamic programming method

The stochastic dynamic programming method (SDP) permits breaking down the inter-temporal optimisation problem into T single stage problems. The problem consists in solving the recursive relation:

$$V_t(x_t) = max_{u_t} \left\{ E\left[r_t(x_t, u_t, k_t)\right] + r \;\; V_{t+1}\left(x_t + g_t(x_t, u_t, k_t)\right) \right\} \qquad t = T, T-1, \ldots, 1 \qquad (25)$$

subject to:

$$V_{T+1}(x_{T+1}) = F(x_{T+1}) \qquad (26)$$

$$\sum_k p_t(k_t) = 1 \qquad (27)$$

$$x(1) = x_1 \qquad (28)$$

As in the deterministic case, the recursive relation permits solving the problem by starting with the last stage and working backwards, stage by stage, to the initial stage. One of the great advantages of dynamic programming is that it permits treating deterministic and random processes similarly.

Applications of this technique to agriculture decision problems have been reviewed by Taylor (1993).

3.2.2. The discrete stochastic programming method

Discrete stochastic programming (DSP) can be used to process sequential decision-making problems in discrete time with a finite horizon when the state and control variables are continuous. This approach was developed by Cocks (1968) and then Rae (1971a).

The DSP method requires that the problem be formulated as a problem of constrained optimisation. Equations are solved simultaneously by using a mathematical programming algorith. Although the notation of stochastic programming models is complicated, they are relatively simple conceptually. The logic of this technique can be understood from the formulation of a model with two decision stages:

$$\text{Maximize} \quad \sum_k p_k \left[r_1(x_1, u_1) + r \;\; r_{2k}(x_{2k}, u_{2k})\right] \qquad (29)$$

subject to $x_{2k} - x_1 = g(x_1, u_1)$ (30)

$u_1, u_{2k} \geq 0$ (31)

where sub-indices 1 and 2 represent the two decision stages, k the state of nature, and p_k the vector of probabilities of states of nature.

This formulation of the model implies that the agent takes several initial decisions (u_1) with uncertain knowledge of the future. This is followed by one of the states of nature (k) and the agent will take other decisions (u_{2k}) later on that depend on the decisions made in the first stage and the state of nature having occurred.

Discrete stochastic programming models have been used by Rae (1971b) to model decision-making in agriculture. They are very flexible and do not require the utility function to be separable; moreover, they permit considering the different sources of risk that influence the objective function and the constraints. However, they are often very large and need considerable amounts of data, thus few DSP applications exist to date. See Apland and Hauer (1993) for a review of the applications of this method.

4. RECURSIVE STOCHASTIC PROGRAMMING: A NEW METHOD OF SOLVING DYNAMIC PROBLEMS?

In practice, the methods for solving the inter-temporal optimisation problems mentioned above suffer from major limitations. Despite the existence of powerful algorithms capable of tackling these problems, the model's variables and/or stages must always remain small in number.

Despite the fact that a large number of decision stages can be considered using the dynamic programming method, since the multi-stage problem in question is broken down into several one-stage problems, the number of state and control variables must remain limited. In practice, this technique requires limiting the possible values of the model's state and control variables to a discrete set. The solutions obtained are therefore approximate and the degree of precision will depend on the differences between the values inside the discrete set. In the case of non-linear functions, the errors can be non-negligible. Furthermore, all the decisions made in the current stage must lead to a "possible" state of the system in the following stage, sometimes requiring that other approximations be made.

Undoubtedly, the most serious disadvantage of dynamic programming is the difficulty of considering the diversity of activities and constraints specific to the field of agricultural and natural resource economics.

On the contrary, discrete stochastic programming permits simultaneously taking into account the uncertainty and the diversity of activities and constraints specific to agricultural decision problems. DSP permits working with continuous variables and non-linear functions[5]. Nonetheless, its application remains limited to problems with a low number of stages. Since optimisation is inter-temporal, the model's size increases exponentially with the number of decision stages.

In the previous models, the decision-maker makes decisions by taking into account their consequences on the future. This entails inter-temporal optimisation under uncertainty and irreversibility. It can be likened to a game of chess: the player takes into account the possible reactions of his adversary and his own counter-reactions in full knowledge of the rules.

This leads us to raising the question of whether the rules are as well known in natural resource economics, i.e. does the agent have full knowledge about the possible responses of nature?

By making the hypothesis that the decision-maker is perhaps more *myopic* than the dynamic programming would like, we propose another method of solving dynamic problems. The main difference of this method in comparison to the previous ones is the way the information enters the problem. In this case, the decision-maker does not have all the information available when making decisions; hence he is unable to fully anticipate the responses of nature and must opt for a sub-optimal decision. Once the first decision has been carried out, the system evolves (the decision-maker knows the response of nature) and the agent can adjust later decisions according to the new information available.

The method consists in solving the dynamic problem by making a series of sequential optimisations, thus it is a recursive method where each optimisation comprises a dynamic model.

Consequently, at moment *1*, the decision-maker chooses a decision plan by taking into account all the information available at this moment. At moment 2, the decision taken for the first stage (u_1) has already been carried out and, as a function of the state of nature happened, the system will have progressed to reach state x_{2k}. The agent can now revise the decision plan, not for stage *1* but for the following stages depending on the new information available. This procedure is illustrated by the following diagram:

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5 Obtaining a global maximum cannot always be achieved by using available non-linear programming algorithms, though it can be obtained by adequate formulation of the problem.

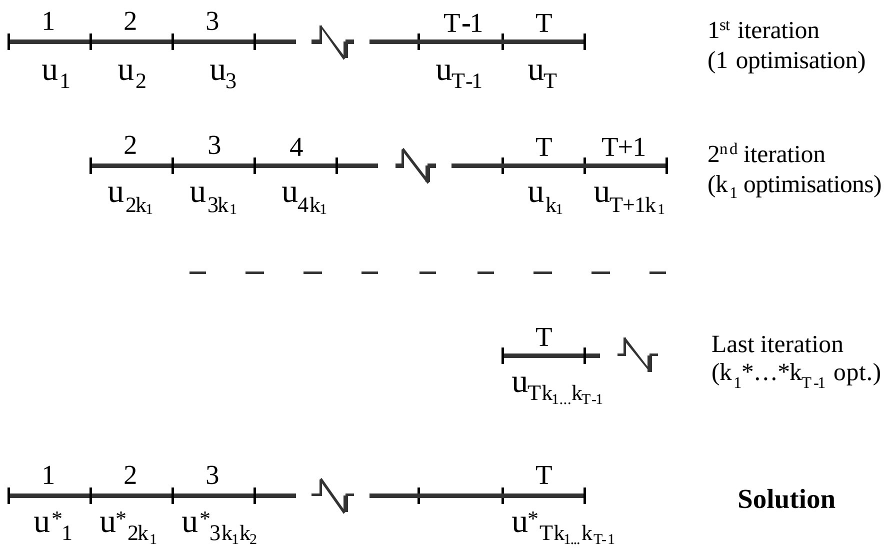

Figure 2. Diagram of the recursive stochastic programming method

The first iteration therefore consists in solving the optimisation problem given by equations:

$$\text{Maximize} \quad \sum_{t=1}^{T} r^{\,t-1}\, r_t(x_t, u_t) \quad + \quad r^T\, F(x_{T+1}) \tag{32}$$

$$\text{subject to} \quad x_{t+1} - x_t = g_t(x_t, u_t) \qquad t = 1,2,\ldots,T \tag{33}$$

$$x(1) = x_1 \tag{34}$$

In this case, function $g_t(x_t,u_t)$ does not depend on the state of nature happened, rather it has a definite value resulting, for example, from taking into account the mathematical expectation of random variable k.

Once the solution has been obtained, we will only take into account the result for the first stage, u_1, and we will determine the state of the system in the following stage for each state of nature k:

$$x_{2k_1} - x_1 = h_1(x_1, u^*{}_1, k_1) \qquad k_1 = 1,2,\ldots K \tag{35}$$

The second iteration consists in solving a series of optimisations, one for each initial state of system x_{2k}:

$$\text{Maximize} \quad \sum_{t=2}^{T+1} r^{\;t-1}\, r_t(x_{tk_1}, u_t) \quad + \quad r^{\,T}\, F(x_{Tk_1+1}) \qquad \forall k_1 \tag{36}$$

$$\text{subject to} \quad x_{t+1,k_1} - x_{tk_1} = g_t(x_{tk_1}, u_t) \qquad t = 1,2,\ldots,T \tag{37}$$

$$x(2) = x_{2k_1} \qquad \forall k_1 \tag{38}$$

This process is repeated T times and the solution to the problem is obtained by retaining the result for the first stage at each iteration:

$$u^* = (u^*{}_1, u^*{}_{2k}, u^*{}_{3k_1k_2}, \ldots) \tag{39}$$

As will be seen in what follows, in the case of simple models this technique gives the same results as the two previous ones, while permitting taking into account a large number of variables and decision stages. However, the sequence of optimal decisions in more complex problems will be different.

This method has major advantages when the system must be represented by a considerable number of state variables or in the case of a large number of possible activities (reservoir management, irrigation management, soil erosion, etc.). Furthermore, it allows the introduction of exogenous changes other than stochastic resource availability.

This type of model permits a sequential representation of decision-making by assuming that decisions are irreversible, as in sequential decision stochastic models. A decision tree can be modelled similar to that used in sequential decision stochastic models. This type of model permits getting round the *curse of dimensionality* and solve a problem with many variables and decision stages.

Several applications of recursive models with multi-stage components have been described in the literature (Louhichi et al., 1999; Cacho, 1998 and 1999). However, the objective of recursivity in these models is not to represent the sequential stochastic nature of the problem, but to permit exogenous changes of some of the model's parameters. What is original in this work is that it proposes using recursive programming as a method for solving sequential stochastic problems.

We could go even further and consider a problem with two different decision horizons: a short-term horizon and a long-term one. We can, for instance, introduce a more thorough modelling of the nearest stages and reduce details as distance increases through time.

To illustrate this procedure, imagine that we wish to model an agricultural decision process in a context of climatic uncertainty. We can assume that long-term decisions (e.g., investment decisions) are taken according to the probability of the occurrence of states of nature. Nonetheless, the farmer can make adjustments (amount of fertiliser, irrigation, etc.) throughout the year. To model this behaviour, we can build a multi-stage model whose first stage of simulation is divided into several sub-stages. Decisions throughout the year are taken sequentially as a function of the state of nature occurring. Investment decisions are taken at the beginning as a function of the probabilities of states of nature throughout the planning horizon.

Since we have to repeat this procedure by using a sliding planning horizon, the model is formulated to adjust the decisions taken for the following years.

5. A NUMERICAL EXAMPLE

Comparison between the different methods of solving dynamic optimisation problems can be shown by using the crop-irrigation problem proposed by Kennedy (1986). In this example, a farmer produces three horticultural crops each year in successive seasons, i.e. each crop occupies the soil for three months. The yield of each crop (y_t) depends on the depth (cm) of water applied (w_t) according to relation $y_t = w_t - 0.1\ w_t^2$. The farmer has a small reservoir for irrigation whose stocks of water vary as a function of consumption and rainfall during each season. The depth of water applied to each crop depends on the water released from storage at the beginning of each season (u_t, in metres) and the rainfall occurring during this season (q_t), i.e. $w_t = u_t + q_t$.

The maximum level of water in the reservoir is 3 metres and it is assumed to be full at the beginning of the year. The amount of water which can be released at the beginning of each season is limited do integer values (metres of water) and by the amount in storage.

The farmer seeks to determine water release in each season (u_t) so as to maximise the present value of receipts from sale of the crops. Thus it is a dynamic problem with three decision stages, where the water used in each stage (u_t) is the control variable while the water stock (x_t) is the state variable.

We approach the deterministic problem first before going on to the stochastic version in which the rainfall of each season is a random variable.

5.1. The deterministic dynamic model

As mentioned above, the farmer seeks to determine the quantity of water released from the reservoir in each stage (u_t) in order to maximise the current value of the farm's revenue (V_1). Since b_t is the price of the crop corresponding to stage t, the revenues of the farm in this stage (r_t) are:

$$r_t = \ 0.1\ b_t \left[u_t + q_t - 0.1 (u_t + q_t)^2 \right] \tag{40}$$

Given that ρ is the discount factor, the problem is written as:

$$\text{Maximize} \quad \sum_{t=1}^{3} r^{t-1}\ r_t \tag{41}$$

$$\text{subject to} \quad 0 \le\ u_t \ \le\ x_t \ \le\ 3 \qquad u_t, x_t \ \text{integers} \tag{42}$$

$$x_{t+1} = min\{ (x_t - u_t + q_t), 3 \} \tag{43}$$

$$x_1 =\ 3 \tag{44}$$

given that b = [50, 100, 150]

q = [2, 1, 1]

r = 0.95

This problem can be solved indifferently by dynamic programming and mathematical programming.

By using dynamic programming, we can express the backward recursive relation as:

$$V_t(x_t) = \max_{0 \le u_t \le x_t} \{ r_t +\ r\ V_{t+1}(x_{t+1}) \} \qquad t = 3,2,1$$

And the problem consists in solving this equation with the conditions:

$$x_{t+1} - x_t = - u_t + q_t$$

$$V_4(x_4) = 0$$

By starting with the last stage:

$$V_3(x_3) = \max_{u_3}\ r_3 = \max_{u_3} \left\{ 0.1\ b_3 \left[u_3 + q_3 - 0.1 (u_3 + q_3)^2 \right] \right\}$$

we can determine the optimal decision u^*_3 and value $V_3(x_3)$ for the different possible values of x_3. Then, on the basis of equation:

$$V_2(x_2) = \max_{u_2} \{r_2 + r\, V_3(x_3)\} = \max_{u_2} \left\{ 0.1\, b_2 \left[u_2 + q_2 - 0.1\,(u_2 + q_2)^2 \right] + r\, V_3(x_3) \right\}$$

we obtain u^*_2 and $V_2(x_2)$ for the different possible values of x_2. Lastly, equation:

$$V_1(x_1) = \max_{u_1} \{r_1 + r\, V_2(x_2)\} = \max_{u_1} \left\{ 0.1\, b_1 \left[u_1 + q_1 - 0.1\,(u_1 + q_1)^2 \right] + r\, V_2(x_2) \right\}$$

allows us to determine u^*_1 and $V_1(x_1)$ for the different possible values of x_1. This resolution procedure leads to the following results:

Decision stage t	State variable x_t	Control variable u_t	Current value $V_t(x_t)$
1	3	2	60.4
2	3	2	50.9
3	2	2	31.5

Table 1. Results of the deterministic problem

This procedure allows us to determine the optimal decision path for other initial water levels.

The same results can be obtained by using a non-linear programming algorithm[6].

5.2. Stochastic dynamic model

Consider now a stochastic version of the crop-irrigation problem introduced in last section. Let us suppose that rainfall in each stage, which influences both crop yields and water stocks in the reservoir, is a random variable (*q*) and that three states of nature can be distinguished (*k*):

State of nature	Decision stage					
	1		2		3	
k	$p_1(k_1)$	q_1^k	$p_2(k_2)$	q_2^k	$p_3(k_3)$	q_3^k
1	0.25	1	0.25	0	0.25	0
2	0.50	2	0.50	1	0.50	1
3	0.25	3	0.25	2	0.25	2

Table 2. Distribution of rainfall probability

6 All the models have been solved by using the GAMS software.

Note that expected rainfall in each stage is the same as for the deterministic problem.

In this case, for each stage *t*, the revenue from the farm depends on the state of nature occurring in stage *t-1*:

$$r_{tk} = 0.1\, b_t \left[u_t + q_{tk} - 0.1 (u_t + q_{tk})^2 \right] \qquad (45)$$

Supposing that the objective function is the expected discount value, the problem can be written as:

$$\text{Maximize} \quad \sum_{t=1}^{3} r^{t-1} \left[\sum_{k=1}^{3} p_{tk}\; r_{tk} \right] \qquad (46)$$

$$\text{subject to} \quad 0 \le u_t \le x_t \le 3 \qquad u_t, x_t \text{ integers} \qquad (47)$$

$$x_{t+1} - x_t = - u_t + q_{kt} \qquad (48)$$

$$x_1 = 3 \qquad (49)$$

This problem can be solved by backward induction by using stochastic dynamic programming, or as an constrained inter-temporal optimisation problem by using discrete stochastic programming. Further on we comment on these methods and on resolution by recursive stochastic programming.

5.2.1. Resolution by stochastic dynamic programming

In this case, we simply need to write the backward recursive relation:

$$V_t(x_t) = \max_{0 \le u_t \le x_t} \left\{ \sum_{k=1}^{3} p_{tk}\; r_{tk} + r\; V_{t+1}(x_{t+1}) \right\} \qquad t = 3,2,1$$

with the transition equation and boundary conditions:

$$x_{t+1} - x_t = - u_t + q_{tk}$$

$$V_4(x_4) = 0$$

$$x_1 = 3$$

Since the final value is known, it is possible to determine the optimal decisions in the third stage for each possible value of x_3 from:

$$V_3(x_3) = \max_{u_3} \sum_{k=1}^{3} p_{3k}\ r_{3k}$$

which allows us to determine the optimal decisions in the second stage by solving:

$$V_2(x_2) = \max_{u_2} \left\{ \sum_{k=1}^{3} p_{2k}\ r_{2k} + r\ V_3(x_3) \right\}$$

and the decisions in the first stage by repeating this process. Now, starting from the system's initial state ($x_1 = 3$), we can obtain the sequence of optimal decisions for each stage as a function of the state of nature occurring.

Stage	Less favourable			More favourable			More probable		
t	Rainfall q_t	State x_t	Control u_t	Rainfall q_t	State x_t	Control u_t	Rainfall q_t	State x_t	Control u_t
1	1	3	2	3	3	2	2	3	2
2	0	2	1	2	3	2	1	3	2
3	0	1	1	2	3	3	1	2	2
Current value		31.2			69.1			60.4	

Table 3. A few results of the stochastic problem (dynamic programming)

Obviously, the objective function optimal value will be less in the stochastic case (57,1) than in the deterministic formulation of the problem (60,4). Table 3 shows the results for the series of "less favourable", "more favourable" and "more probable" rainfalls.

5.2.2. Resolution by discrete stochastic programming

We shall now solve the example as an constrained inter-temporal optimisation problem.

Notation becomes more complex, because the state and control variables depend on the states of nature occurring in the past; the model is not solved recursively but by an optimisation algorithm. In our example, although the decision to be taken in the first stage (u_1) does not depend on states of nature, that of the second stage (u_{2k}) will depend on the state of nature occurring in the first stage, while that of the third stage (u_{3km}) will depend on the states of nature in the two previous stages.

To simplify notation, we designate the possible states of nature in stage 1 by k, those of stage 2 by m and those of stage 3 by n. The formulation of the discrete stochastic programming problem requires differentiating the state and control variables for each decision stage. In our example:

Decision stage	State variables	Control variables
1	x_1	u_1
2	x_{2k}	u_{2k}
3	x_{3km}	u_{3km}
4	x_{4kmn}	

The farm's revenues in each stage are:

$$r_{1k} = 0.1\, b_1\, (u_1 + q_{1k} - 0.1\,(u_1 + q_{1k})^2) \tag{50}$$

$$r_{2km} = 0.1\, b_2\, (u_{2k} + q_{2m} - 0.1\,(u_{2k} + q_m)^2) \tag{51}$$

$$r_{3kmn} = 0.1\, b_3\, (u_{3km} + q_{3n} - 0.1\,(u_{3km} + q_{3n})^2) \tag{52}$$

For each possible path (for each branch of the decision tree), the current value of the farm's revenues will be:

$$VA_{kmn} = r_{1k} + r\ \ r_{2km} + r^2\ r_{3kmn} \tag{53}$$

Since we attempt to maximise the expected present value, the problem is written as:

$$\text{Maximize} \quad \sum_{k=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3} p_k\ p_m\ p_n\ VA_{kmn} \tag{54}$$

$$\text{subject to} \quad x_{2k} = x_1 - u_1 + q_{1k} \tag{55}$$

$$x_{3km} = x_{2k} - u_{2k} + q_{2m} \tag{56}$$

$$x_{4kmn} = x_{3km} - u_{3km} + q_{3n} \tag{57}$$

$$u_1 \le x_1 \le 3; \quad u_{2k} \le x_{2k} \le 3; \quad u_{3km} \le x_{3km} \le 3 \tag{58}$$

$$x_1 = 3 \tag{59}$$

with non-negativity conditions for the variables.

The results obtained with this method are the same as those obtained with the dynamic programming method. The main difference is that dynamic programming permits obtaining the optimal sequence of decisions for any initial state of the system, whereas discrete stochastic programming only gives the solution for $x_1 = 3$:

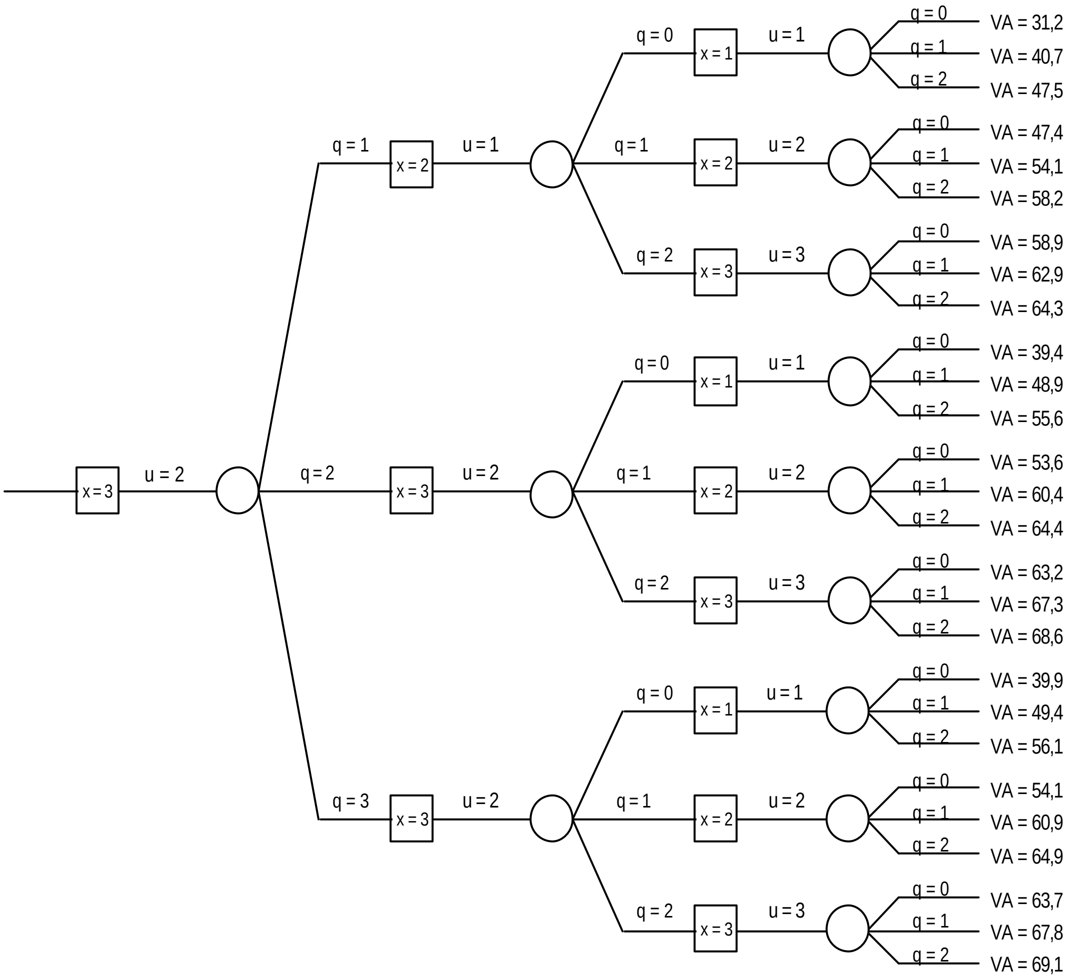

Figure 3. Decision tree of the stochastic problem

5.2.3. Resolution by recursive stochastic programming

This method consists in solving a series of inter-temporal optimisation problems.

In the first iteration, we make the hypothesis that the agent reasons in terms of expected rainfall values. Thus we solve the problem by:

$$\text{Maximize} \quad \sum_{t=1}^{3} r^{t-1}\, r_t \qquad (60)$$

$$\text{subject to} \quad 0 \leq u_t \leq x_t \leq 3 \qquad u_t, x_t \ \text{integers} \qquad (61)$$

$$x_{t+1} = min\{(x_t - u_t + q_t), 3\} \qquad (62)$$

$$x_1 = 3 \qquad (63)$$

Once the first decision has been carried out (u^*_1), uncertainty related to the rainfall of the first stage will be cleared (one of the possible states of nature q_k will occur), which will affect both the revenues generated r^*_{1k} and the state of the system at the beginning of the second decision stage (x^*_{2k}). These relations are given by the following recursivity equations:

$$r^*_{1k} = 0.1\ b_1 \left[u^*_1 + q_{1k} - 0.1\,(u^*_1 + q_{1k})^2\right] \tag{64}$$

$$x^*_{2k} = x_1 - u^*_1 + q_{1k} \tag{65}$$

The agent will find himself in one of the states of nature x^*_{2k} with a probability p_k instead of finding himself in the state expected x_2. The agent can revise his decisions for the following stages as a function of the state reached for the system (which will depend both on the decisions taken and on the rainfall during the stage).

The second iteration permits determining the optimal decisions throughout the remaining planning horizon given the state of the system and the expected rainfall values. The second iteration therefore consists in solving the dynamic problem shifted by one stage for each initial value x^*_{2k}:

$$\text{Maximize} \quad \sum_{t=2}^{3} r^{\,t-1}\ r_{tk} \tag{66}$$

$$\text{subject to} \quad 0 \leq u_{tk} \leq x_{tk} \leq 3 \qquad u_{tk}, x_{tk} \ \text{integers} \tag{67}$$

$$x_{t+1,k} = \min\{\,(x_{tk} - u_{tk} + q_t)\,,\,3\,\} \tag{68}$$

$$x_{2k} = x^*_{2k} \tag{69}$$

We shall now only take into account the results obtained for the second stage (u^*_{2k}) and determine the revenues generated r^*_{2km} and the state of the system at the beginning of the following stage (x^*_{3km}) as a function of the state of nature having occurred (m):

$$r^*_{2km} = 0.1\ b_1 \left[u^*_{2k} + q_{2m} - 0.1\,(u^*_{2k} + q_{2m})^2\right] \tag{70}$$

$$x^*_{3km} = x_{2k} - u^*_{2k} + q_{2m} \tag{71}$$

The third iteration permits determining the optimal decisions throughout the remaining planning horizon given the state of the system (x^*_{3km}) and the expected rainfall values (cf. Figure 4).

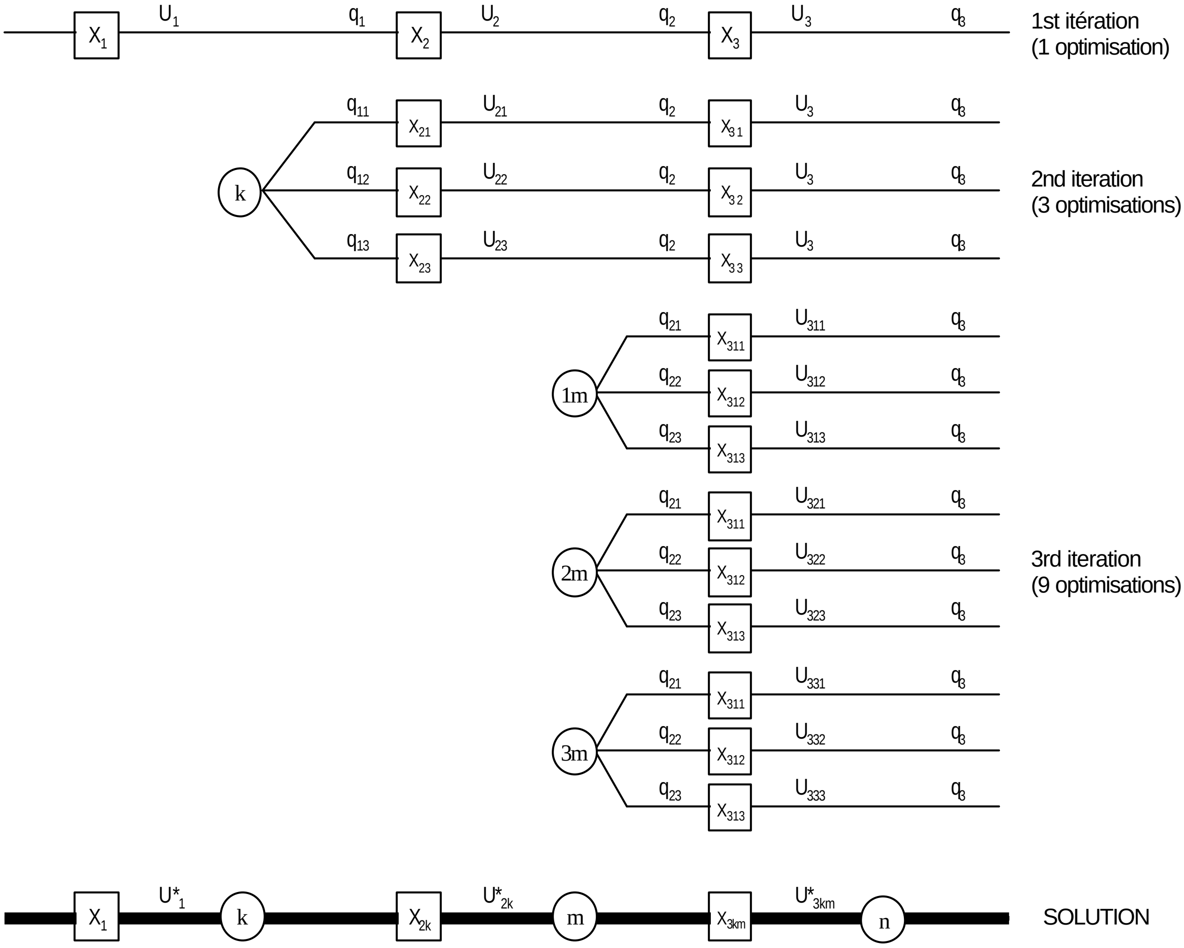

Figure 4. Diagram of the resolution by recursive stochastic programming

In our example, the results obtained with recursive stochastic programming coincide with the results obtained with the previous methods. This is not, whatever the circumstances, the general case since the problem's information structure is different. In fact, when the solution to the stochastic problem is obtained by using continuous and non-discrete state and control variables, the results are not the same. The discrete stochastic programming model gives an expected value of 57.258, whereas the recursive stochastic programming model gives 57.18. Nonetheless, it should be borne in mind that this value is close to that obtained with discrete variables (57.1).

On reaching this point, the reader may ask, "What is the point of using this "forward" recursive method instead of the "backward" recursive method? The advantage of this method is that it does not require using discrete values for state and control variables and it allows us to incorporate a large number of variables and equations into the model. We try to clear up this point in the next section.

6. A SLIGHTLY MORE COMPLEX EXAMPLE

In what follows we present a slightly more complex example. It also entails modelling inter-temporal allocation decisions concerning irrigation water in a context of stochastic water availability.

The problem is to determine the optimal price of irrigation water for an irrigation district so that farmers can take into account the value of the resource's scarcity. The water comes from a reservoir whose annual replenishment is highly variable and uncertain. Obviously, the demand for water in any year cannot exceed the water available in the reservoir during this stage. If the demand for water is less than its availability, the water is stored in the reservoir and can be used in future years.

To simplify the problem, we represent the irrigation district by a single representative farm. In each crop year, the producer allocates the available surface and water to a number of crops so as to maximise a certain objective function (profit). We assume that all the model's coefficients are known and that only replenishment of the reservoir with water is random (see Blanco, 1999, for a detailed explanation).

6.1. Non-sequential model

We can model the reservoir's inter-annual management by using a non-sequential model. In this case, we suppose that all the decisions are taken at the same time.

Given the system's objective function and constraints, the farmer attempts to determine the optimal decisions of production, investment in irrigation technology and water use in each decision stage.

The main *decision variables* are production activities (vector X_{crit}) and activities of investment in irrigation technology (vector Y_{rt}). Each production activity is defined as a crop (*c*) with an irrigation technique (*r*) and an water allocation (*i*) during one year (*t*). The irrigation technology investment activities correspond to the purchase of irrigation equipment for the farm.

The problem is expressed algebraically as follows:

$$\text{Maximize} \quad NPV = \sum_t \frac{Z_t}{(1+ta)^{t-1}} \quad ; \quad t = 1, \cdots, 12 \tag{72}$$

$$\text{subject to} \quad \sum_c \sum_r \sum_i X_{crit} \leq S \qquad \forall t \tag{73}$$

$$\sum_c \sum_i e_{cr} X_{crit} - N_{rt} \leq 0 \qquad \forall t \tag{74}$$

$$N_{r,t+1} = N_{rt} + Y_{rt} - A_{rt} \qquad \forall t \tag{75}$$

$$\sum_c \sum_r \sum_i q_{cri} X_{crit} = Q_t \; h \; ; \quad Q_t \leq D_t \qquad \forall t \tag{76}$$

$$D_{t+1} = D_t - Q_t + R_t \qquad \forall t \tag{77}$$

$$N_{rt}(t=1) = n_{r1} \; ; \; D_t(t=1) = D_1 \tag{78}$$

$$X_{crit} \geq 0; \quad Y_r \geq 0 \quad Q_t \geq 0 \tag{79}$$

NPV: net present value
Z_t: total income of the irrigated farm during stage t (ptas),
ta: discount rate
X_{crit}: surface allocated to each production activity (ha),
S: total surface area of the farm (ha),
e_{cr}: requirements for irrigation equipment r of activity c,
N_{rt}: level of irrigation equipment of type r (ha),
Y_{rt}: surface area equipped for irrigation with technique r (ha) during stage t,
q_{cri}: water requirement per production activity c (m^3/ha),
Q_t: water released from the reservoir during stage t (m^3),
R_t: replenishment of the reservoir during stage t (m^3),
D_t: water availability at the beginning of stage t (m^3)
h: water distribution efficiency coefficient.

The model maximises the net present value (equation 72) subject to constraints on land (equation 73), irrigation equipment (equation 74), water use (equation 76), transition (equations 75 and 77) and initial conditions (equation 78). The model includes other activities and constraints (financing, crop rotations, market constraints, etc.) that we have obviate to simplify the presentation.

Although replenishment of the reservoir is a random variable (R), we can introduce this uncertainty on water availability in a non-sequential model in two ways:

- ➔ By supposing that the farmer reasons in terms of average replenishment. In this case, the value of annual replenishment (R_t) will be given by the expected value of random replenishment (R):

$$R_t = E[R]$$

- ➔ By supposing that the farmer takes his decisions in order to guarantee a certain level of security. In this case, we can use chance constrained programming, thus the annual replenishment (Rt) to be considered in the model will be:

$$R_t = E[R] - K_a \; s_R$$

For example, if the farmer decides the production plan that can be carried out in at least 90% of cases, the annual replenishment value will be equal to replenishment representing 90% security:

$$R_t = E[R] - 1.28 \, s_R$$

The non-sequential model (deterministic or chance constrained) does not permit modelling inter-annual management decisions. Since replenishment with water is constant for all the years, these models provide a static solution in which the quantity released every year is equal to the replenishment value.

6.2. Sequential model

A sequential formulation must be used if we wish to model water storage decisions in order to overcome situations of scarcity.

Specifying the information structure is an essential step in formulating the stochastic sequential decision problem. In our example, the producer takes irrigation technology investment decisions and surface allocation to the crops before having complete information on water availability. Once the availability of water is known, later on, it will be possible to make certain adjustments (the farmer can decide on the quantities of water to be allocated to the crops).

The only source of uncertainty that we take into consideration is water reservoir replenishment. We define states of nature as different levels of replenishment of the reservoir (which reproduce the probability distribution) with associated probabilities, and we suppose that the probability of occurrence of each state of nature is independent from the state occurring in the previous stage. To simplify the problem, we have only considered three states of nature.

Consequently, at the beginning of the year, the farmer will decide the production plan as a function of his expectations of water availability. Once the availability of water is known, he can adjust the allocations of water to the crops.

The decision tree of this problem is similar to that of figure 3 though the number of decision stages has now been increased to 12 (12 years). The number of the tree's branches will therefore be 3^{12}, i.e. 531441. Consequently, solving this problem by discrete stochastic programming is impractical. It can be solved by dynamic programming, but achieving this requires discretizing the values of variables Q, D and R and solving thousands of optimisation problems to determine a profit function depending on water use.

Another method would be recursive stochastic programming. In this case, we suppose that the farmer reasons in the long term (investment decisions) as a function of the expected reservoir water replenishment values, whereas, in the short-term, the farmer can make adjustments (allocations of water to crops).

In practice, the method involves dividing the first year into two decision steps: in the first step, the farmer decides the investments and the production plan; in the second step once the replenishment value is known, the farmer adjusts the allocations of water to the crops.

The first iteration consists in solving:

$$\text{Maximize} \quad E[Z] = \sum_k p_k \left(\sum_t \frac{Z_{tk}}{(1+ta)^{t-1}} \right) \tag{80}$$

$$\text{subject to} \quad \sum_c \sum_r X1_{cr} \leq S \quad si \;\; t=1 \tag{81}$$

$$\sum_c \sum_r \sum_i X2_{critk} \leq S \tag{82}$$

$$\sum_c \sum_i e_{cr} X2_{critk} - N_{rt} \leq 0 \tag{83}$$

$$N_{r,t+1} = N_{rt} + Y_{rt} - A_{rt} \tag{84}$$

$$\sum_c \sum_r \sum_i q_{cri} X2_{critk} = Q_{tk}\, h \;\; ; \quad Q_{tk} \leq D_{tk} \tag{85}$$

$$D_{t+1,k} = D_{tk} - Q_{tk} + R_{tk} \tag{86}$$

$$N_{rt}(t=1) = n_{r1} \;\; ; \;\; D_t(t=1) = D_1 \tag{87}$$

$$X_{critk} \geq 0; \quad Y_r \geq 0 \quad Q_{tk} \geq 0 \tag{88}$$

where:

Z_{tk}: the total income of the irrigated area in stage t and state of nature k (ptas),

p_k: probability of occurrence of state of nature k,

$X1_{cr}$: surface area allocated to each production activity (ha) in the first decision step,

$X2_{critk}$: surface area allocated to each production activity (ha) in the second decision step,

Q_{tk}: quantity of water released from the reservoir in state of nature k (m^3),

D_{tk}: water availability in state of nature k (m^3),

R_{kt}: reservoir replenishment in state of nature k (m^3),

Once the solution to this problem is obtained, we take into account the result of the first stage and use the relations of recursivity to determine the initial state of the system for the second iteration:

$$N_{rt}(t=2) = N_{r1}^* + Y_{r1}^* - A_{r1}^* = N_{r2}$$

$$D_{tk}(t=2) = D_1 - Q_{1k}^* + R_{1k}^* = D_{2k}$$

Afterwards we solve the model with a planning horizon shifted by one stage, by taking as starting point the initial values N_{r2} and D_{2k}. In the second iteration, we obtain the decisions to allocate resources in the second stage.

We repeat this process to obtain the optimal decisions for the remaining stages.

This approach has two advantages in comparison to non recursive methods:

- A practical advantage, since it avoids the curse of dimensionality, which occurs when using both discrete stochastic programming and dynamic programming.

- A methodological advantage, in terms of using a more realistic representation of the decision-making process, because with this approach we take a less "clear-cut" view of the future. Our decision-maker is not a "champion chess-player" as is implied by dynamic programming or discrete stochastic programming.

7. REFERENCES

APLAND, J. AND HAUER, G. (1993). *Discrete stochastic Programming: Concepts, Examples and a Review of Empirical Applications*. Staff Paper P93-21, Departement of Agricultural and Appliel Economics, University of Minnesota.

BARBIER, B. (1998). Induced innovation and land degradation. Results from a bioeconomic model of a village in West Africa. *Agricultural Economics*, 19 (1-2): 15-25.

BARBIER, B. AND BERGERON, G. (1999). Impact of policy interventions on land management in Honduras: results of a bioeconomic model. *Agricultural Systems*, 60: 1-16.

BELLMAN, R.E. (1957). *Dynamic Programming*. Princenton University Press, Princenton, New Jersey, USA.

BLANCO FONSECA, M. (1999). *Economía del agua: Análisis de políticas de modernización de regadíos en España*. Tesis Doctoral. Uiversidad Politécnica de Madrid. Madrid.

CACHO, O.J. (2000). The Role of Bioeconomic Models in Renewable Resource Management and Assessment of Solution Techniques. Paper presented at the *XXIV International Conference of Agricultural Economics (IAAE)*, Berlin, Agosto 2000.

DAY, R.H. (1961). Recursive Programming and Supply Predictions. In E.D. Heady et al. (eds.): *Agricultural Supply Fonctions*, Iowa State University Press, Ames, Iowa, pp. 108-125.

DAY, R.H. AND CIGNO, A. (EDS.) (1978). *Modeling Economic Change: The Recursive Programming Approach*, North-Holland Publishing Company, Amsterdam.

KENNEDY, J.O.S. (1986). *Dynamic Programming. Applications to Agriculture and Natural Resources*. Elsevier.

LOUHICHI, K.; FLICHMAN, G. AND ZEKRI, S. (1999). Un modèle bio-économique pour analyser l'impact de la politique de conservation des eaux et du sol. *Economie Rurale*, 252: 55-64:

RAE, A.N. (1971a). Stochastic programming, utility, and sequential decision problems in farm management. *American Journal of Agricultural Economics*, 53, pp. 448-460.

RAE, A.N. (1971b). An empirical application and evaluation of discrete stochastic programming in farm management. *American Journal of Agricultural Economics*, 53, pp. 625-638.

STANDIFORD, R.B. AND HOWITT, R.E. (1992). Solving empirical bioeconomic models: a rangeland management application. *American Journal of Agricultural Economics*, 74, pp. 421-433.

TAYLOR, C.R. (1993). *Applications of Dynamic Programming to Agricultural Decision Problems*. Westview Press, Inc., Boulder, Colorado, USA.

WILEN, J.E. (1985). Bioeconomics of Renewable Resource Use. In A.V. Kneese et J.L. Sweeney (eds.): *Handbook of Natural Resources and Energy Economics*, vol 1, North Holland, Amsterdam, pp. 61-124.

YATES, C.M. AND REHMAN, T. (1998). A Linear Programming Formulation of the Markovian Decision Process Approach to Modelling the Dairy Replacement Problem. *Agricultural Systems*, 58 (2): 185-201.